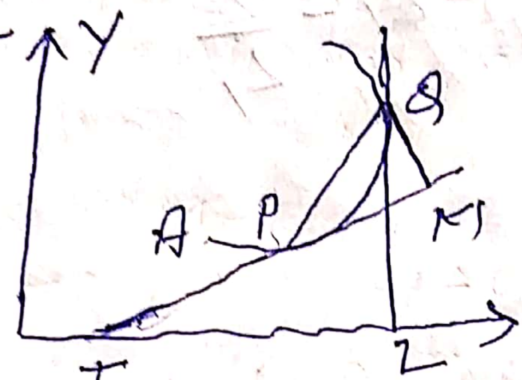
B.Sc (Math) part II paper IV

Components of velocity and Acceleration (Tangential and Normal)

Theorem :- A particle is moving in a plane curve, find the components of its velocity along the tangent to the curve at any instant.

proof:-



Let A be the fixed point on the curve and P, Q the position of the particle at time t and $t+\delta t$ respectively.

Let the tangent at P and Q to the curve be PT and QL. Let arc $AP = s$ and $AQ = s+\delta s$ so that $PQ = \delta s$. We draw $QM \perp$ to the tangent at P.

Tangential velocity at P

$$= \lim_{\delta t \to 0} \frac{[\text{Displacement along TP at time } t+\delta t] - [\text{Displacement along TP at time } t]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PM - 0}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{chord } PQ)\cos \angle QPM}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} \cdot \cos \angle QPM$$

$$= 1 \cdot \frac{ds}{dt} \cdot 1 \quad \text{as } \delta t \to 0 \ \cos \angle QPM = 1$$

and $\frac{\text{chord } PQ}{\delta s} \to 1$

$$= \dot{s}$$

Normal velocity at P

$$= \lim_{\delta t \to 0} \frac{[\text{displacement perpendicular to TP in time } t+\delta t] - [\text{displacement perpendicular to TP in time } t]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{QM - 0}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{chord } PQ) \sin \angle QPM}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} \sin \angle QPM$$

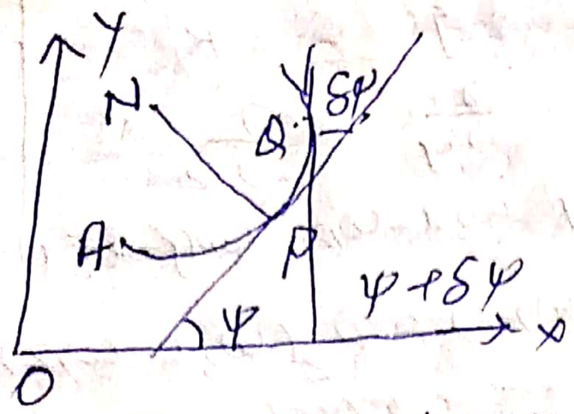
$$= 1 \cdot \frac{ds}{dt} \cdot 0 \quad \text{as } \sin \angle QPM \to 0 \text{ as } \delta t \to 0$$

$$= 0$$

Hence the normal velocity $= 0$

Theorem :- A particle is moving in a plane curve, find the components of the acceleration along the tangent and normal to the curve at a point

proof :- Let A be the fixed point on the curve and P and Q the positions of the particle at time t and $t+\delta t$ respectively



Let the tangent at P and Q makes angles ψ and $\psi+\delta\psi$ with the axis x so that $\delta\psi$ is the angle between the tangents.

Let arc $AP = S$ and $AQ = S+\delta S$ so $PQ = \delta S$

Let v and $v+\delta v$ be the ~~velocity~~ velocities of the particle at P and Q along the tangent at P and Q.

Now tangential acceleration at P

$$= \underset{\delta t \to 0}{Lt} \frac{[\text{velocity along } TP \text{ at time } t+\delta t] - [\text{velocity along } TP \text{ at } t]}{\delta t}$$

$$= \underset{\delta t \to 0}{Lt} \frac{(v+\delta v)\cos\delta\psi - v}{\delta t}$$

$$= \underset{\delta t \to 0}{Lt} \frac{(v+\delta v)\cdot 1 - v}{\delta t}$$ as $\delta\psi$ is small so we may write $\cos\delta\psi = 1$

$$= \underset{\delta t \to 0}{Lt} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$= \dfrac{d^2s}{dt^2}$ and also $\dfrac{dv}{ds}\cdot\dfrac{ds}{dt}$

$$= v\frac{dv}{ds}$$

Hence the tangential velocity

$= \frac{dv}{dt}$ ie $\frac{d^2s}{dt^2}$ ie $v\frac{dv}{ds}$

Normal acceleration of P

$= \underset{\delta t \to 0}{Lt}$ [velocity perpendicular to TP at time $t+\delta t$] - [velocity perpendicular to TP at time t] / δt

$= \underset{\delta t \to 0}{Lt} \frac{(v+\delta v)\sin\delta\varphi - 0}{\delta t}$

$= \underset{\delta t \to 0}{Lt} \frac{(v+\delta v)\delta\varphi}{\delta t}$ as $\delta\varphi$ is small

$= \underset{\delta t \to 0}{Lt} \frac{v\delta\varphi}{\delta t}$, neglecting $\delta v \delta\varphi$ is small

$= v\frac{d\varphi}{dt} = v\frac{d\varphi}{ds}\cdot\frac{ds}{dt} = \frac{v^2}{\rho}$

as $\frac{ds}{dt} = v$ and $\rho = \frac{ds}{d\varphi}$

Hence normal acceleration

$= \frac{v^2}{\rho}$